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Conservative numerical methods for solitary wave interactions

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Abstract
The purpose of this paper is to show the advantages that represent the use of numerical methods that preserve invariant quantities in the study of solitary wave interactions for the regularized long wave equation. It is shown that the so-called conservative methods are more appropriate to study the phenomenon and provide a dynamic point of view that allows us to estimate the changes in the parameters of the solitary waves after the collision.

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1. Introduction

The interaction of solitary waves was first studied in connection with the Korteweg–de Vries equation (KdV) (see [1]),

$$u_t + u_x + (u^2)_x + u_{xxx} = 0 \qquad -\infty < x < \infty \qquad t > 0 \tag{1}$$

that was originally derived as a model for the unidirectional propagation of water waves of small amplitude and long wavelength. Equation (1) admits a family of solutions that are solitary waves, and the numerical studies of Zabusky and Kruskal [2] indicated that the result of a nonlinear interaction of a pair of solitary waves leaves them unaltered except for a phase shift. The proof of this soliton property for the KdV was obtained by using the inverse-scattering method [3], a technique that can be applied to partial differential equations that, as KdV, possess an infinite number of conserved quantities. The inverse-scattering method can be viewed in the light of the generalization, to infinite dimension, of the theory of integrable Hamiltonian systems of ordinary differential equations [4].

An alternative model to equation (1) is the regularized long wave equation (RLW),

$$u_t + u_x + (u^2)_x - u_{xxt} = 0 \qquad -\infty < x < \infty \qquad t > 0 \tag{2}$$

initially suggested by Peregrine [5], with physical and computational advantages with respect to (1) (see [6]) and widely studied by Benjamin, Bona and Mahoney [7]. In fact, (2) is also known as the BBM equation. It has solitary wave solutions, similar to those for (1). But, as far as the interaction is concerned, equation (2) presents some remarkable differences. First numerical experiments [6] suggested that the interaction is 'inelastic': after this, the waves emerge not only with a phase shift, but also with a change in their amplitudes. Besides that, the formation of a small 'rarefaction' wave of dispersive nature behind the smaller wave is also observed. The numerical evidence of these differences with respect to the interaction in (1) gave rise to an analytical study of the phenomenon. Olver [8] showed that (2) is not integrable in the sense that it only has a finite number of conserved quantities, which prevents the use of the inverse-scattering technique. On the other hand, in [9], Bona showed that, in physical situations in which the waves are of small amplitude, both (1) and (2) can be used to describe the model, in the sense that the solutions of KdV and RLW equations remain close for times that are not too large. However, the bound of the difference grows with time and the soliton property for (1) may not be a small-amplitude phenomenon. This does not make this result applicable to the study of the interaction of solitary waves for (2). In a similar way, in [10] a different version of the RLW equation (2) was considered as a perturbation of (1). The author used perturbation techniques to obtain an approximation to the solution of the RLW equation from the solution given by the KdV equation, generalizing the perturbation methods for Hamiltonian systems of ordinary differential equations that are approximately integrable [4]. This allowed us to verify that there exists a change in the velocities of the two solitary waves after interaction, for the case considered.

But, in general, solutions of (2) can only be obtained by using numerical simulation [11–14]. These numerical studies, specially [11, 14], focused exclusively on highly accurate numerical methods. We think that accuracy is an important criterion, but it is not the only one; that is, not only the size of the errors matters, but also the direction in which they propagate in time. A good behaviour of the errors is determined by the fact that the numerical integrator used retains qualitative properties of the system of differential equations under consideration [15]. A choice of the numerical method exclusively based on accuracy may provide approximations whose qualitative behaviour is incorrect in relation to the phenomenon being described.

The use of qualitative aspects of the numerical schemes has revealed special interest in the integration of solitary waves. More specifically, for equation (2), the importance of taking into account the conserved quantities of the equation was established in [16]. One of the main conclusions is that the numerical solution of methods that preserve some of these invariants behaves as a solitary wave that basically retains the amplitude of the original one, with a phase shift that grows linearly in time with respect to the original phase. However, in the case of 'nonconservative' methods, the numerical approximation shows an amplitude error which grows with time and a quadratic perturbation in the phase. This behaviour justifies the great interest in numerical methods with conservation properties in order to study the interaction of solitary wave solutions of (2).

The paper is structured as follows. In section 2 we describe the different behaviour of numerical approximations to solitary wave solutions of the RLW equation in connection with conservation properties. Section 3 is devoted to justifying the use of conservative methods for the solitary wave interaction problem. There are two main reasons for this selection: the first one is that numerical approximations obtained with conservative integrators show a better qualitative behaviour before the interaction in order to simulate the collision properly. On the other hand, the use of this kind of methods provides a dynamic point of view that allows us to estimate the parameters of the solitary waves emerging from the interaction.

2. Solitary waves for the RLW equation and numerical methods

First we describe some properties of the RLW equation that will be used in our study. Equation (2) admits only three independent quantities conserved by the solutions, namely [8]

$$C(u) = \int_{-\infty}^{\infty} u \, \mathrm{d}x \tag{3}$$

$$I(u) = \int_{-\infty}^{\infty} \left(\frac{1}{2}u^2 + \frac{1}{2}u_x^2 \right) \mathrm{d}x \tag{4}$$

$$H(u) = \int_{-\infty}^{\infty} \left(\frac{1}{2}u^2 + \frac{1}{6}u^3 \right) \mathrm{d}x \tag{5}$$

that are usually called 'mass' , 'momentum' and 'energy', respectively.

A second property is that (2) possesses a two-parameter family of solitary wave solutions of the form

$$\begin{aligned} &\psi(x, t, c, x_0) = A \operatorname{sech}^2(K(x - ct - x_0)) \\ &A = \frac{3}{2}(c - 1) \qquad K = \frac{1}{2}\left(1 - \frac{1}{c}\right)^{1/2}. \end{aligned} \tag{6}$$

The parameter $c > 1$ determines not only the velocity of the wave but also its amplitude A, with the taller the wave the faster the travel. On the other hand, x_0 governs the initial location of the solitary wave.

The influence of the invariants (3)–(5) in the numerical integration of solitary waves (6) was analysed in [16]. The study was done with semidiscrete (discrete t and continuous x) approximations obtained by one-step integrators with fixed step size for the initial value problem for (2) of the form

$$U^{n+1} = \phi_{\Delta t}(U^n). \tag{7}$$

Here Δt denotes the time step, $U^n = U^n(x)$ is a numerical solution at time level $t_n = n\Delta t, n = 0, 1, \ldots,$ and $\phi_{\Delta t}$ is a mapping that approximates the flow of the equation. Thus, if $U^0 = u_0$, then U^n represents an approximation to the value $u(t_n)$ of the solution u of (2) with initial condition u_0.

The analysis of the asymptotic expansion of the approximation to a solitary wave (6) states that, under not restrictive hypotheses about (7), if $U^0 = \psi(x, 0, c, x_0)$, then the numerical solution $U^n(x)$ can be written as (see [16, 17])

$$U^n(x) = \psi(x, t_n, \tilde{c}, \tilde{x}_0) + (\Delta t)^r \rho(x, t_n) + (\Delta t)^r Q(x, t_n, \Delta t) \tag{8}$$

where

$$\tilde{c} = c + \alpha_2 t_n (\Delta t)^r \tag{9}$$

$$\tilde{x}_0 = x_0 + \left(\alpha_1 t_n + \alpha_2 \frac{t_n^2}{2} \right) (\Delta t)^r \tag{10}$$

for some constants α_1, α_2 and where r is the order of the method (7). Now we make some comments on formula (8).

(i) The first term on the right-hand side of (8) is a solitary wave of the form (6) but with new parameters satisfying equations (9)–(10). We observe that the velocity $\tilde{c}$ of this modified wave (resp. amplitude) is a linear-in-time perturbation of the velocity c of the original wave (resp. amplitude). On the other hand, $\tilde{x}_0$ differs from the original x_0 in terms that grow quadratically with time, and so does the phase of the modified wave, $\tilde{x}_0 + \tilde{c}t_n$ with respect to the original $x_0 + ct_n$. An important fact to point out (see [16]) is that if the numerical method (7) preserves the quantity (4) or the energy (5), then $\alpha_2 = 0$ in (9)–(10). Therefore, the modified wave maintains the original velocity (and consequently the amplitude) while the phase only differs from the original one in terms that grow linearly with time.

(ii) The second term $(\Delta t)^r \rho$ on the right-hand side of (8) represents errors of leading order $O((\Delta t)^r)$ not associated with changes in the parameters of the wave. In general, this term may grow at most linearly with time, but in the case where the numerical method conserves the quantity (3) and one of the other two (4) or (5), it remains bounded [16].

(iii) Finally, the third term in (8), $(\Delta t)^r Q$, is a remainder of higher order $o((\Delta t)^r)$ for fixed t. Although this term may grow with time, it is numerically controlled for reasonably long times (see also [17]).

Therefore, formula (8) says that the modified solitary wave is a good guide for the numerical solution and determines its behaviour. Consequently, it can be understood that the numerical approximation to a solitary wave of the family (6) given by a method that conserves (4) or (5) (the conservation of (3) is satisfied by practically all standard numerical methods since it is a linear functional [18]) maintains basically a solitary wave profile. Furthermore, it behaves essentially as a wave with a phase shift with respect to the original, but with similar amplitude. This provides a simulation of the solitary wave more appropriate, in a qualitative sense, than that of a general numerical scheme. In this last case, the numerical solution looks like a wave whose amplitude (velocity) separates from the theoretical one linearly with time and whose phase grows quadratically.

3. Numerical simulation

The aim of this section is to disclose the influence of the conservative character of the numerical integrators in the simulation of the solitary wave interaction phenomenon. We focus on two issues of the problem: the evolution of the solitary waves before the interaction and the estimation of the parameters of the solitary waves emerging after collision.

The following standard methods, just considered in [16] for the analysis of one solitary wave integration, are used for this numerical study:

(i) The well-known implicit midpoint rule, a second-order method that preserves the quantity I but not the Hamiltonian H (see [19]), is taken as an example of conservative scheme.

(ii) The nonconservative method selected is the simply diagonally implicit Runge–Kutta method with tableau

$$\begin{array}{|cc} \gamma & 0 \\ 1-2\gamma & \gamma \\ \hline 1/2 & 1/2 \end{array}$$

where $\gamma = (3+\sqrt{3})/6$. This is a third-order method which does not preserve the quantity I nor the Hamiltonian H.

The choice of these methods is mainly determined by their behaviour in long time integrations in connection with their conservation properties. We do not pretend to establish a comparison between the efficiency of the schemes. On the other hand, both methods preserve the first invariant C. We have preferred to use competitive schemes instead of constructing mass nonpreserving integrators. Furthermore, as we pointed out in the previous section, the more important differences in the behaviour of the approximations are stated in relation to the other two conserved quantities I and H. Finally, a method preserving the Hamiltonian H and not the invariant I could have been chosen as an example of conservative integrator, instead of the implicit midpoint rule (as in [16]).

To implement in practice these semidiscrete schemes we first discretize the spatial variable using a Fourier pseudospectral approximation, so that errors virtually correspond to the time integrators [20]. To this end, we consider a spatial interval $0 \leqslant x \leqslant L$ with periodic boundary

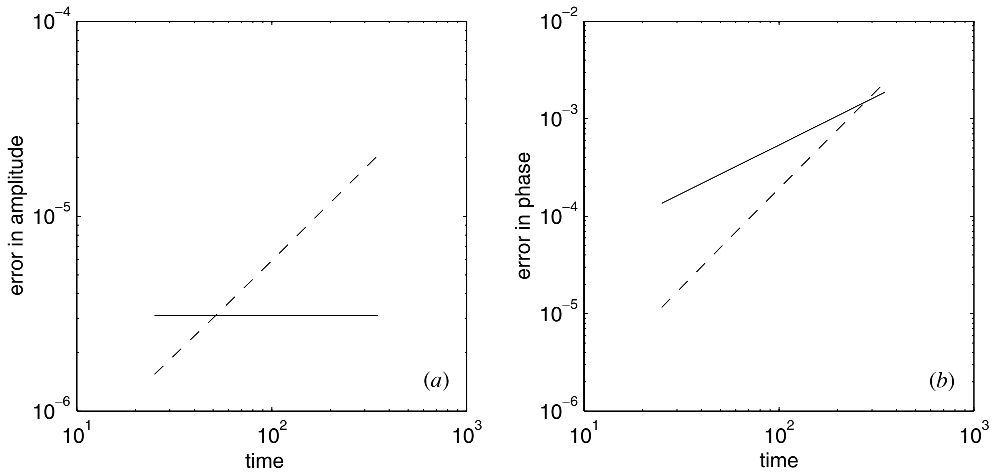

Figure 1. Evolution of the error in the simulation of a single solitary wave: (*a*) shows the evolution of the error in the amplitude, (*b*) shows the evolution of the error in the phase. Solid line (——) corresponds to the conservative method and dashed line (- - - -) to the nonconservative method.

conditions. This interval is chosen sufficiently large to assure that initially the solution is properly represented inside it, in the sense that the value of the solution outside this interval can be rejected due to the zero exponential decay of the solitary waves of the family (6). We achieve a practically exact spatial discretization by successively doubling the number of spatial grid points until a grid was found for which no further error reduction was possible. The use of pseudospectral approximation in spatial discretization is justified by its exponential convergence. Having in mind that we are interested in conservation properties along time integration, other discretizations are also valid while error in space is negligible with respect to error in time.

3.1. Solitary wave evolution

We begin our study by simulating the evolution of a single solitary wave. These experiments illustrate the different behaviour of the numerical methods with respect to the evolution of the numerical profile parameters. This will be used later, when considering the evolution of two waves.

In the first experiment we take a solitary wave of the family (6) with velocity $c = 5$ (then the amplitude is $A = 6$) and initially located at $x_0 = 256$. We integrate the equation by taking as initial condition the restriction of this wave to the spatial grid. We use the same step size, $\Delta t = 10^{-3}$, for both methods. The output obtained in each case consists of the grid values of the numerical solution, so we can compute the amplitude of the corresponding numerical solitary wave and the point when this maximum value is attained (the phase) from the corresponding interpolating trigonometric polynomial.

Figure 1 presents, in logarithmic scale, the evolution in time of the error in the parameters (amplitude and phase) of the numerical solution for both methods. Solid line corresponds to the results obtained with the conservative scheme and dashed line is associated with the nonconservative one.

Figure 1(*a*) shows the evolution of the error in the amplitude. Note that in the conservative case this error remains constant but in the nonconservative case the error growth is linear.

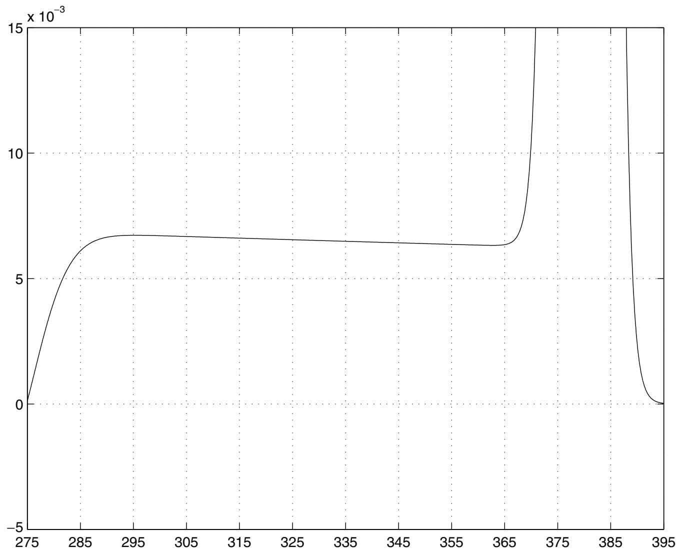

Figure 2. Tail behind the numerical profile obtained with the nonconservative method. The vertical scale is magnified.

Figure 1(*b*) refers to the evolution of the phase error. In both cases, there is a phase shift that grows with time, but, as the slopes of the lines show, the numerical profile obtained with the conservative method separates from the original wave linearly, being quadratically in the nonconservative case.

The same error behaviour can be observed when different step sizes Δt are used, and when different solitary wave velocities are considered.

Finally, the change in amplitude observed in the nonconservative scheme gives rise to another harmful phenomenon related to the first conserved quantity C. We have already noted that this invariant, which represents the area enclosed by the wave, is preserved by the numerical solution given by each integrator. In the case of the nonconservative method selected, the approximation decreases in amplitude with respect to the original one. Thus, the mass lost is recovered through the formation of a tail behind the numerical profile that disfigures the wave form. This tail can be observed in figure 2, which displays the numerical solution in a magnified vertical scale. This phenomenon does not occur in the conservative case.

These results confirm the theoretical analysis stated in [16] and discussed in section 2.

In spite of the nonlinearity of the equation, two solitary waves which do not interact are virtually the sum of two single waves due to the exponential decay of both waves. We have confirmed numerically this point by considering as initial data for the numerical simulation the superposition of two solitary waves. We conclude that the parameters for each solitary wave are just the same, up to the precision being used, as the values computed when each solitary wave is simulated separately. This implies that, when the conservative method is used, each profile shows constant-in-time amplitude and its phase evolves linearly. In the case of the nonconservative scheme, the amplitude decreases with time linearly and the phase evolution is quadratic.

As a first consequence, in the numerical simulation of the solitary wave interaction, the initial location of the waves is crucial when a nonconservative method is used. This is due to

Table 1. Comparison between the numerical parameters and the theoretical ones in the case of a single solitary wave with $A = 6$, $x_0 = 256$, $c = 5$.

Δt	$A_{\Delta t}$	$\lvert A - A_{\Delta t}\rvert$	$x_{0\Delta t}$	$\lvert x_0 - x_{0\Delta t}\rvert$	$c_{\Delta t}$	$\lvert c - c_{\Delta t}\rvert$
8×10^{-3}	5.999 806	1.9×10^{-4}	255.999 884	1.2×10^{-4}	4.999 657	3.4×10^{-4}
4×10^{-3}	5.999 951	4.9×10^{-5}	255.999 971	2.9×10^{-5}	4.999 914	8.6×10^{-5}
2×10^{-3}	5.999 988	1.2×10^{-5}	255.999 993	7.2×10^{-6}	4.999 979	2.1×10^{-5}
1×10^{-3}	5.999 997	3.1×10^{-6}	255.999 998	1.8×10^{-6}	4.999 995	5.4×10^{-6}

the modification of the parameters explained above, specially in the amplitude. If the solitary waves are initially too separated, when they are about to collide, their numerical counterparts would not give a reliable representation of the interaction. This would explain that in the numerical simulations carried out in the literature with this kind of methods, the waves are initially placed sufficiently close [11, 14]. But this is not the case of a conservative method because the evolution in time of the numerical solitary waves does not damage their amplitudes. Therefore the numerical waves remain undisturbed from the beginning, independently of their initial location, and so they represent the interaction accurately . The main wrongness is the collision time, due to the phase shift that both waves suffer.

3.2. Solitary wave interaction

We have shown the importance of using conservative methods in order to simulate solitary waves properly. However, the more appropriate behaviour of conservative integrators also gives a dynamic point of view that allows us to estimate the parameters of the solitary waves from their numerical counterparts. This will be of special interest in the simulation of solitary wave interaction after the collision.

The numerical studies performed in the literature [6, 11, 14] show that, after the collision of two solitary wave solutions of (2), two new profiles appear. But the amplitude and phase of these emergent waves are different from those of the original. Furthermore, a small tail wave of dispersive nature behind the slower solitary wave is also noted. All these properties are observed in our numerical experiments, but our objective is to show how to estimate the parameters of the new solitary waves after interaction by using conservative methods.

As we have seen, the numerical approximation of a solitary wave obtained with a conservative method maintains a wave profile with practically constant amplitude, and phase that evolves linearly with time (see figure 1). Thus, we assign to this numerical wave a numerical amplitude $A_{\Delta t}$ determined by fitting the amplitudes of the numerical solution computed at different time levels. On the other hand, the linear evolution in time of the numerical phase allows us to fit a line to the corresponding results computed at the same output times. Thus, we obtain a numerical initial location $x_{0\Delta t}$, and a numerical velocity $c_{\Delta t}$ of the numerical wave. The parameters $A_{\Delta t}, x_{0\Delta t}$ and $c_{\Delta t}$ computed in this way are approximations to the theoretical ones A, x_0 and c, respectively.

To illustrate this convergence, in table 1 we present these numerical parameters computed, with the conservative method, for the simulation of a single solitary wave with $c = 5$ ($A = 6$) and $x_0 = 256$ and by fitting the output obtained at $t = 25, 50, 75, \ldots, 300$. Note that the error columns in the table show a second-order convergence of the numerical parameters to the theoretical ones. This is the order of the numerical method. Similar results are obtained for simulations of solitary waves with other parameters and, taken into account the remarks in subsection 3.1, we get the same numerical parameters when the simulation includes another

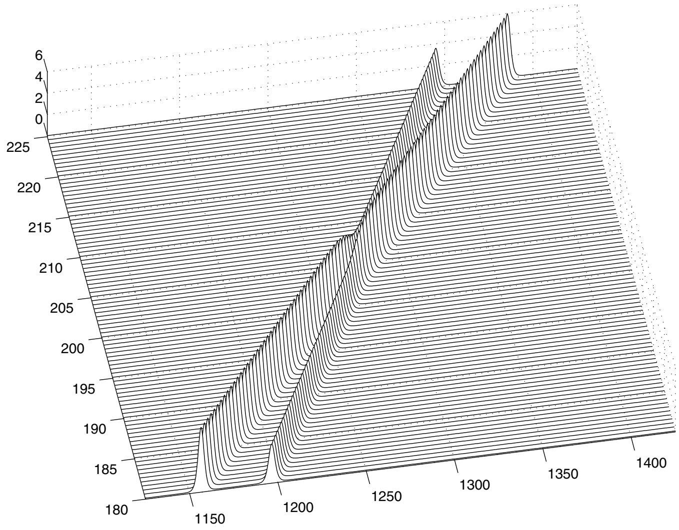

Figure 3. Perspective plot of the collision of two solitary waves with a conservative method.

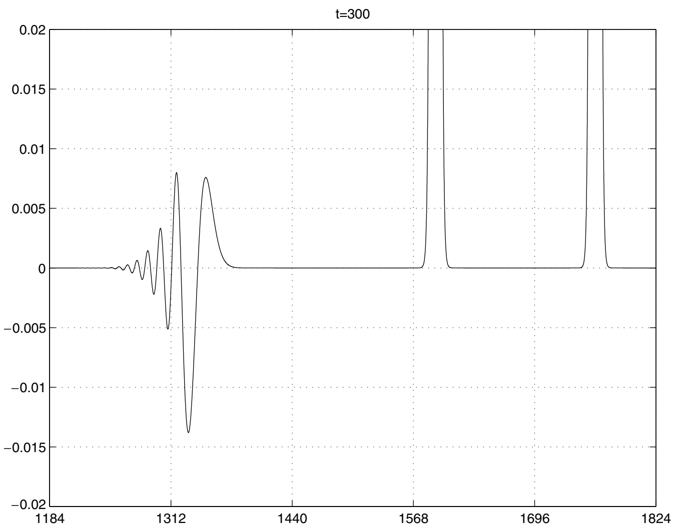

Figure 4. Dispersive tail appearing after the collision of two solitary waves.

solitary wave but without the interaction. This behaviour allows us to estimate the parameters of the theoretical solitary wave being studied.

As an application of this numerical procedure, we can consider an experiment of wave interaction. We take the superposition of two solitary waves as initial data. The parameters are $c^{(1)} = 5$ ($A^{(1)} = 6$), $x_0^{(1)} = 256$ for the first wave, and $c^{(2)} = 3.4$ ($A^{(2)} = 3.6$), $x_0^{(2)} = 576$ for the second one.

Figure 3 shows a perspective plot of the collision of the two solitary waves. As in other works, we observe that, two new numerical solitary waves emerge from the interaction. Besides that, a more careful look (see figure 4, which displays the numerical solution after the collision in a magnified vertical scale) shows the formation of a tail. We have checked that the

Table 2. Numerical parameters of the faster solitary wave evolved in the interaction. Initially $A^{(1)} = 6, x_0^{(1)} = 256, c^{(1)} = 5$.

	$A_{\Delta t}^{(1)}$		$x0_{\Delta t}^{(1)}$		$c_{\Delta t}^{(1)}$	
Δt	Before	After	Before	After	Before	After
8×10^{-3}	5.999 806	5.999 917	255.999 884	259.958 953	4.999 657	4.999 732
4×10^{-3}	5.999 951	6.000 063	255.999 971	259.958 720	4.999 914	4.999 989
2×10^{-3}	5.999 988	6.000 099	255.999 993	259.958 661	4.999 979	5.000 053
1×10^{-3}	5.999 997	6.000 109	255.999 998	259.958 647	4.999 995	5.000 069

Table 3. Numerical parameters of the slower solitary wave evolved in the interaction. Initially $A^{(2)} = 3.6, x_0^{(2)} = 576, c^{(2)} = 3.4$.

	$A_{\Delta t}^{(2)}$		$x0_{\Delta t}^{(2)}$		$c_{\Delta t}^{(2)}$	
Δt	Before	After	Before	After	Before	After
8×10^{-3}	3.599 943	3.599 643	575.999 942	571.480 233	3.399 910	3.399 710
4×10^{-3}	3.599 986	3.599 686	575.999 985	571.480 530	3.399 977	3.399 778
2×10^{-3}	3.599 996	3.599 697	575.999 996	571.480 604	3.399 994	3.399 795
1×10^{-3}	3.599 999	3.599 700	575.999 999	571.480 622	3.399 999	3.399 799

number of oscillations of this tail increases with time and its maximum amplitude decreases, suggesting a dispersive nature.

As previous works indicate, the parameters of both solitary waves change due to the collision, although this is undistinguishable in figure 3. We can estimate this change as follows: from the evolution of these profiles we calculate their numerical parameters $A_{\Delta t}^{(i)}, x0_{\Delta t}^{(i)}$ and $c_{\Delta t}^{(i)}, i = 1, 2$, by using the technique described above. In tables 2 and 3 we present these parameters computed for both numerical waves before and after the interaction. By using the numerical parameters $x0_{\Delta t}^{(i)}$ and $c_{\Delta t}^{(i)}, i = 1, 2$, we can compute the numerical phase of each solitary wave in any established time.

First of all, note that from table 2, the faster wave increases its amplitude/velocity, while in table 3 we observe a decrease in amplitude of the slower wave. This behaviour coincides with that of previous works. Furthermore, there is a phase shift in the waves, as the variation in the parameters $x0_{\Delta t}^{(1)}, x0_{\Delta t}^{(2)}$ suggests.

Now, taking into account the second-order convergence of the numerical parameters, we use extrapolation with the results corresponding to the step sizes 2×10^{-3} and 10^{-3}. Thus, we can estimate that the faster wave resulting from the interaction has velocity $c^{(1)} = 5.000\,074$ ($A^{(1)} = 6.000\,111$) and would be initially placed at $x_0^{(1)} = 259.958\,642$ if it was alone. For the slower one, $c^{(2)} = 3.399\,800$ ($A^{(2)} = 3.599\,701$) and $x_0^{(2)} = 571.480\,628$. Hence, the faster solitary wave increases its velocity in 7.4×10^{-5} (and its amplitude in 1.1×10^{-4}), but the slower one decreases its velocity in 2.0×10^{-4} (and its amplitude in 3.0×10^{-4}).

As a conclusion, the experiments performed in this paper reveal that the use of conservative methods shows great advantages in the evolution of solitary wave simulations for the RLW equation. This is particularly important in the study of the interaction phenomenon, where no analytical solution is available. This kind of methods would provide an appropriate starting point in the numerical analysis of other more complicated equations with solitary wave solutions and conserved quantities.

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References

[1] Whitham G B 1974 *Linear and Nonlinear Waves* (New York: Wiley)
[2] Zabusky N J and Kruskal M D 1965 *Phys. Rev. Lett.* **15** 240–3
[3] Miura R M 1974 *Nonlinear Waves* ed S Leibovitch and A R Seebass (Ithaca, NY: Cornell University Press)
[4] Arnold V I 1989 *Mathematical Methods of Classical Mechanics* (Berlin: Springer)
[5] Peregrine D H 1966 *J. Fluid Mech.* **25** 321–30
[6] Abdulloev K O, Bogolubsky I L and Makhankov V G 1976 *Phys. Lett.* A **56** 427–8
[7] Benjamin T B, Bona J L and Mahoney J J 1972 *Phil. Trans. R. Soc.* A **272** 47–78
[8] Olver P J 1979 *Math. Proc. Camb. Phil. Soc.* **85** 143–60
[9] Bona J L 1978 *Proc. Int. Congress of Mathematicians* (Providence, RI: American Mathematical Society) p 865
[10] Byatt-Smith J G B 1987 *J. Fluid Mech.* **182** 467–83
[11] Bona J L, Pritchard W G and Scott L R 1980 *Phys. Fluids* **23** 438–41
[12] Eilbeck J C and McGuire G R 1977 *J. Comput. Phys.* **23** 63–73
[13] Gardner L R T and Gardner G A 1990 *J. Comput. Phys.* **91** 441–59
[14] Sloan D M 1991 *J. Comput. Appl. Math.* **36** 159–79
[15] Sanz-Serna J M 1997 *The State of the Art in Numerical Analysis* ed I S Duff and G A Watson (Oxford: Clarendon) 121–43
[16] Araújo A and Durán A 2001 *Appl. Numer. Math.* **36** 197–217
[17] Durán A and López-Marcos M A 2002 *Appl. Numer. Math.* **42** 95–116
[18] Gear C W 1990 *Physica* D **60** 303–10
[19] Sanz-Serna J M and Calvo M P 1994 *Numerical Hamiltonian Problems* (London: Chapman and Hall)
[20] Tadmor E 1986 *SIAM J. Numer. Anal.* **23** 1–10